

For the general case of submodular value functions, the expected value of the set function $v(\cdot)$ is upper bounded by its concave closure (Definition 3) as follows. The allocation rule $\mathbf{x}(\cdot)$ that optimizes this virtual surplus program induces, for $\mathbf{c} \sim \mathbf{F}$, a distribution over sets of winning agents. Denote this distribution by \mathcal{D} and denote by $\hat{\mathbf{q}}$ the profile of marginal probabilities, i.e., with $\hat{q}_i = \Pr_{S \sim \mathcal{D}}[i \in S]$. By the definition of the concave closure of the set function $v(\cdot)$, $\mathbf{E}_{\mathbf{c}}[v(\mathbf{x}(\mathbf{c}))] = \mathbf{E}_{S \sim \mathcal{D}}[v(S)] \leq V^+(\hat{\mathbf{q}})$.

The payment to an agent is lower bounded by the payment from price posting. As above, the optimal mechanism selects agent i with probability \hat{q}_i . When virtual costs are monotonically increasing, i.e., in the case of *regular distributions*, the expected payment to an agent i selected with probability \hat{q}_i is minimized if agent i is served if and only if $c_i \leq F_i^{-1}(\hat{q}_i)$ by Lemma 1 since these costs minimize $\phi_i(\mathbf{c})$.³ Thus, the mechanism that minimizes expected payments and serves each agent i with probability \hat{q}_i is the mechanism that posts price $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ to each agent i .

LEMMA 3. *For any agent with cost drawn from regular distribution F_i and any incentive compatible mechanism that selects agent i with probability \hat{q}_i , the expected payment of agent i is at least $\hat{q}_i \hat{c}_i$ where $\hat{c}_i = F_i^{-1}(\hat{q}_i)$.*

Combining the relaxation of the value function and the relaxation of the payments we obtain the following concave closure program,

$$\begin{aligned} \max_{\mathbf{q}} \quad & V^+(\mathbf{q}) \\ \text{s.t.} \quad & \sum_i q_i F_i^{-1}(q_i) \leq B. \end{aligned} \quad (4)$$

LEMMA 4. *Let $\hat{\mathbf{q}}^+$ be the optimal solution to the concave closure program (4), then $V^+(\hat{\mathbf{q}}^+)$ upper bounds the performance of the optimal ex ante mechanism in the case of regular cost distributions.*

Posted price mechanisms are trivially incentive compatible. Since the distributions of agents' costs are independent, the set of agents who will accept their offer with a posted price mechanism is a set which will contain each agent with some probability q_i independently. Therefore the performance of a posted price mechanism where agents accept their offer with probabilities \mathbf{q} is the multilinear extension $V(\mathbf{q})$. This motivates us to rewrite the concave closure program (4) as the following multilinear extension program,

$$\begin{aligned} \max_{\mathbf{q}} \quad & V(\mathbf{q}) \\ \text{s.t.} \quad & \sum_i q_i F_i^{-1}(q_i) \leq B. \end{aligned} \quad (5)$$

Maximizing the multilinear extension program gives us an ex ante posted price mechanism that is approximately optimal.

THEOREM 3. *In the case of monotone submodular value functions and regular cost distributions, the ex ante mechanism that posts price $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ to each agent i is an $1 - 1/e$ approximation to the optimal ex ante mechanism, where $\hat{\mathbf{q}}$ is the optimal solution to the multilinear extension program (5).*

³The case of irregular distributions is considered in Appendix B.

PROOF. Let $\hat{\mathbf{q}}^+$ be the optimal solution to the concave closure program (4). Then, by Theorem 1, $V(\hat{\mathbf{q}}^+) \geq (1 - 1/e)V^+(\hat{\mathbf{q}}^+)$. By the optimality of $\hat{\mathbf{q}}$, $V(\hat{\mathbf{q}}) \geq V(\hat{\mathbf{q}}^+)$. Since the performance of posting price $F_i^{-1}(\hat{q}_i)$ to each agent i is $V(\hat{\mathbf{q}})$ and since $V^+(\hat{\mathbf{q}}^+)$ upper bounds the performance of the optimal ex ante mechanism by Lemma 4, posting price $F_i^{-1}(\hat{q}_i)$ to each agent is an $1 - 1/e$ approximation to the optimal ex ante mechanism. \square

Note that in the additive case where each agent has value v_i , $V(\mathbf{q}) = V^+(\mathbf{q}) = \sum_i v_i q_i$ and we get the following corollary.

COROLLARY 1. *In the case of additive value functions and regular cost distributions, the ex ante mechanism that posts price $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ to each agent i is an optimal mechanism, where $\hat{\mathbf{q}}$ is the optimal solution to the multilinear extension program (5).*

We discuss the computational issues of finding a good solution \mathbf{q} to the multilinear extension program (5) in Section 6. For the case of submodular functions, we reduce the problem to submodular function maximization (with a cardinality constraint) for which the greedy algorithm gives an $1 - 1/e$ approximation. In the additive case, we will show that the optimal ex ante budget feasible mechanism can be found by taking the Lagrangian relaxation of the virtual surplus program (3).

4. SUBMODULAR VALUE AND OBLIVIOUS POSTED PRICING

In the previous section, we obtained an ex ante mechanism by optimizing the multilinear extension program (5). In this section we analyze the performance of oblivious posted pricing (with an ex post budget constraint).

The approach of this section is the following: lower the budget by some small amount and optimize the multilinear extension program (5) so that the lowered budget is satisfied ex ante. With the budget sufficiently lowered, with high probability the cost (sum of prices) of the set of agents who would accept their offer is under the original budget (regardless of their arrival order and ex post).

This approach is a special case of that taken by the contention resolution schemes of Vondr ak et al. [26] and we first review some known bounds. The first comes from the submodularity of the value function; the second comes from the Chernoff bound.

THEOREM 4. [Bansal et al. [5]] *Given a non-negative monotone submodular function $v(\cdot)$, a random set R which contains each agent i independently with probability \hat{q}_i , and a (possibly randomized) procedure π that maps (possibly infeasible) sets to feasible sets such that,*

- (marginal property) for all i ,

$$\Pr_{R \sim \hat{\mathbf{q}}; \pi}[i \in \pi(R) \mid i \in R] \geq \gamma,$$

and

- (monotonicity property) for all $T \subseteq S$ and $i \in T$,

$$\Pr_{\pi}[i \in \pi(T)] \geq \Pr_{\pi}[i \in \pi(S)],$$

then $\mathbf{E}_{R \sim \hat{\mathbf{q}}; \pi}[v(\pi(R))] \geq \gamma \cdot \mathbf{E}_{R \sim \hat{\mathbf{q}}}[v(R)]$.

THEOREM 5. [Vondrák et al. [26]⁴] Given $\epsilon \in (0, 1/2)$, budget B , independent variables p_i that are the payments to each agent such that,

- (scaled ex ante budget constraint) $\sum_i \mathbf{E}[p_i] \leq (1 - \epsilon)B$,
- (k -large market) p_i is bounded by $[0, B/k]$ for all i , and
- $k > 2/\epsilon$,

then the probability that the sum of costs of selected agents does not exceed the budget less the cost of any agent, i.e., $\Pr[\sum_i p_i \leq (1 - 1/k)B]$, is at least $1 - e^{-\epsilon^2(1-\epsilon)k/12}$.

We now connect these two results by relating the probability that the sum of costs does not exceed $(1 - 1/k)B$ of Theorem 5 to γ of Theorem 4 and then show that posted pricings satisfy the conditions of Theorem 4.

LEMMA 5. For sequential posted pricing (\hat{c}, σ) that satisfy the scaled ex ante budget constraint and k -large market conditions, the probability that an agent is offered her price is lower bounded by $\Pr_{R \sim \hat{q}}[\sum_{i \in R} \hat{c}_i \leq (1 - 1/k)B]$, the probability that the sum of the prices of agents who would accept their offered price is at most $(1 - 1/k)B$.

PROOF. If the total cost of all agents who would accept their price is at most $(1 - 1/k)B$ then this budget remains at the time an agent i is considered in the sequence σ . By the definition of $k \geq B/\hat{c}_i$ it is feasible to serve this agent and so she is offered her price \hat{c}_i by the sequential posted pricing mechanism. \square

LEMMA 6. For sequential posted pricing (\hat{q}, σ) , if each agent is offered her price with probability at least γ , then the expected value of the mechanism is at least $\gamma V(\hat{q})$.

PROOF. It suffices to show, for sequential posted pricing (\hat{q}, σ) with an ex post budget constraint B , that the marginal and monotonicity properties of Theorem 4 hold.

In our case, $R \sim \hat{q}$ is the random set of agents who would accept their offer if the budget never runs out. Given a set of agents R who accept their offer, define $\pi(R)$ to be the set of agents who accept their offer and who arrive before the budget runs out. In our case, π is deterministic given the ordering σ . Note that $\Pr_{R \sim \hat{q}; \pi}[i \in \pi(R) \mid i \in R]$ is equal to the probability that an agent gets offered her price, meaning that she arrives before the budget runs out. Thus, by the assumption of the lemma the marginal property holds.

For the monotonicity property, consider two sets $T \subseteq S$. When an agent i arrives in the posted price mechanism, the mechanism has spent less if the set of agents who accept their offer is T than if this set is S . Therefore $i \in \pi(S)$ implies that $i \in \pi(T)$ and the monotonicity property holds. \square

By combining the previous results, we obtain the main theorem for this section.

THEOREM 6. For $\epsilon \in (0, 1/2)$, if the oblivious posted pricing \hat{c} corresponding to the optimal solution \hat{q} to the multilinear extension program (5) with budget $(1 - \epsilon)B$ (i.e., with $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ for each agent i) satisfies $2/\epsilon \leq k \leq B/\max_i \hat{c}_i$, then this posted pricing mechanism is a $(1 - 1/e)(1 - \epsilon)(1 - e^{-\epsilon^2(1-\epsilon)k/12})$ approximation to the optimal

⁴The formulation of this theorem is slightly different than in [26] but follows easily from their analysis.

mechanism for submodular value functions and $(1 - \epsilon)(1 - e^{-\epsilon^2(1-\epsilon)k/12})$ for additive value functions in the case of regular cost distributions.

PROOF. The proof starts with the ex ante mechanism from the previous section and then applies results from this section to modify it into an ex post mechanism.

Let \hat{q} be the optimal solution to the multilinear extension program (5) with budget $(1 - \epsilon)B$, $\hat{q}_{(1-\epsilon)B}^+$ be the optimal solution to the concave closure program (4) with budget $(1 - \epsilon)B$, and \hat{q}_B^+ be the optimal solution to the concave closure program (4) with budget B .

By the optimality of \hat{q} and Theorem 1,

$$V(\hat{q}) \geq V(\hat{q}_{(1-\epsilon)B}^+) \geq (1 - \frac{1}{e})V^+(\hat{q}_{(1-\epsilon)B}^+).$$

Note that the solution $(1 - \epsilon)\hat{q}_B^+$ has cost at most $(1 - \epsilon)B$ since $F_i^{-1}(\cdot)$ is increasing. So by the optimality of $\hat{q}_{(1-\epsilon)B}^+$ and by the concavity of the concave closure $V^+(\cdot)$,

$$V^+(\hat{q}_{(1-\epsilon)B}^+) \geq V^+((1 - \epsilon)\hat{q}_B^+) \geq (1 - \epsilon)V^+(\hat{q}_B^+).$$

Since $V^+(\hat{q}_B^+)$ is an upper bound on the performance of the optimal ex ante mechanism by Lemma 4, the ex ante posted pricing mechanism defined for each agent by $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ is a $(1 - 1/e)(1 - \epsilon)$ approximation to the optimal mechanism.

We now consider the posted pricing mechanism defined by \hat{c} that is no longer ex ante. Since the budget has been lowered by a factor $1 - \epsilon$, each agent is offered her price with probability at least $\Pr_{R \sim \hat{q}}[\sum_{i \in R} \hat{c}_i \leq (1 - 1/k)B]$ by Lemma 5, regardless of the ordering σ of agents. By Theorem 5, this probability is at least $1 - e^{-\epsilon^2(1-\epsilon)k/12}$. Therefore, by Lemma 6, the expected value of this mechanism is at least $(1 - e^{-\epsilon^2(1-\epsilon)k/12})V(\hat{q})$ and this mechanism is a $(1 - \epsilon)(1 - 1/e)(1 - e^{-\epsilon^2(1-\epsilon)k/12})$ approximation to the optimal mechanism in the case of submodular value functions. In the case of additive functions, there is no loss from the multilinear extension to the concave closure, so the mechanism is a $(1 - \epsilon)(1 - e^{-\epsilon^2(1-\epsilon)k/12})$ approximation. \square

Note that as the size of the market k grows to infinity, this approximation ratio approaches $1 - 1/e$. Also note that this mechanism requires the market to be at least 4-large. Using another result from Vondrák et al. [26] and a similar analysis to the one from this section, a $(1 - 1/e)/8$ posted pricing mechanism can easily be obtained for any market size. This posted pricing attains its performance guarantee when agents with cost at least $B/4$ arrive before all others, but otherwise the order is oblivious.

5. ADDITIVE VALUE AND SEQUENTIAL POSTED PRICING

In this section we give improved bounds for sequential posted pricing, i.e., where the mechanism orders the agents, and when the value function is additive, i.e., $v(S) = \sum_{i \in S} v_i$. In particular, we analyze the sequential posted pricing (\hat{c}, σ) with $\hat{c}_i = F_i^{-1}(\hat{q}_i)$ from the solution to the multilinear extension program (5) with the full budget B and the ordering σ by decreasing bang-per-buck, i.e., v_i/\hat{c}_i for agent i .

Our results in this section are based on the analysis of the correlation gap of fractional and integral-knapsack set functions (to be defined subsequently). The fractional-knapsack set function is a submodular function, so a correlation gap of

$1-1/e$ can be directly obtained (Theorem 1). In this section, we improve this bound to $1 - 1/\sqrt{2\pi k}$ for k -large markets, i.e., with $k = B/\max_i \hat{c}_i$. From this bound we observe that the correlation gap for fractional-knapsack in large market is asymptotically one. We show that the integral-knapsack correlation gap is nearly the same. Following the approach of Yan [27], the factor by which sequential posted pricing approximates the ex ante relaxation is equal to the integral-knapsack correlation gap.

Definition 5. The *fractional-knapsack* set function corresponding to additive set function $v(S) = \sum_{i \in S} v_i$, sizes \hat{c} , and capacity B is denoted $v_B(S)$ and equals the maximum value solution to the corresponding fractional-knapsack problem on elements S .⁵ The *integral-knapsack* set function can be defined analogously to the fractional one, but it cannot add elements fractionally.

Most of this section analyzes the ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ (see Definition 3 for the definition of independent and correlated values) in the case where the budget constraint is met ex ante, i.e., $\mathbf{E}_{S \sim \hat{q}}[v_B(S)]/\mathbf{E}_{S \sim \mathcal{D}}[v(S)]$ when $\sum_i \hat{c}_i \hat{q}_i \leq B$. We then show that this ratio is equal to the approximation ratio of the sequential posted pricing mechanism. Finally, we use this ratio to bound the integral, and fractional, knapsack correlation gap.

The main idea to derive a bound on this ratio is to show that it is minimized when all agents have equal cost B/k , in which case, when the budget constraint is met ex ante, we can then apply the result from Yan [27] for the correlation gap of the k -highest-value-elements set function.

LEMMA 7. *For any additive value function $v(\cdot)$ and budget B , over marginal probabilities \hat{q} and prices \hat{c} that (a) satisfy the ex ante budget constraint, i.e., $\sum_i \hat{c}_i \hat{q}_i \leq B$, and (b) satisfy the k -large market condition, i.e., $\hat{c}_i \leq B/k$, the ratio of the independent value of the fractional-knapsack and the correlated value of $v(\cdot)$ is minimized when $\hat{c}_i = B/k$ for all i .*

PROOF. For the first part of the proof, we assume that $v = \hat{c}$, i.e., that the bang-per-buck is one for all elements. The last step of the proof is to generalize this special case to any values. Observe that with this assumption, $v_B(S) = \min(B, \sum_{j \in S} \hat{c}_j)$.

Assume that there is some \hat{c}_i such that $\hat{c}_i < B/k$. We show that when $v_i = \hat{c}_i$, increasing \hat{c}_i to any $\hat{c}'_i > \hat{c}_i$ and decreasing \hat{q}_i to $\hat{q}'_i = \hat{c}_i \hat{q}_i / \hat{c}'_i$ preserves the correlated value while only lowering the independent value. Let $\hat{c}'_j = \hat{c}_j$ and $\hat{q}'_j = \hat{q}_j$ for $j \neq i$. The correlated value of $v(\cdot)$ is $\mathbf{E}_{S \sim \mathcal{D}}[v(S)] = \sum_j \hat{c}_j \hat{q}_j = \sum_j \hat{c}'_j \hat{q}'_j$ so it is preserved. Similarly, the ex ante budget constraint is still satisfied.

The argument for the independent value decreasing is the following. Let $v'_B(S)$ be defined similarly as $v_B(S)$, but where agents have values and costs equal to \hat{c}' . Condition on the subset of other agents S who accept their prices and consider the marginal contribution to the expected value of $v_B(\cdot)$ and $v'_B(\cdot)$ from agent i . In the case that $C = \sum_{j \in S} \hat{c}_j > B$, this contribution is zero for both \hat{c}_i and

⁵This value is given by sorting the elements of S by v_i/\hat{c}_i and admitting them greedily until the first element that does not fit with the remaining capacity, that element is admitted fractionally (providing a fraction of its value).

\hat{c}'_i . When $C < B$, these contributions are $\hat{q}_i \min(B - C, \hat{c}_i)$ and $\hat{q}'_i \min(B - C, \hat{c}'_i)$. By the definition of $\hat{q}'_i = \hat{c}_i \hat{q}_i / \hat{c}'_i$ and concavity of $\min(B - C, \cdot)$, the former is greater than the latter. This inequality holds for all sets S , so removing the conditioning on S , it holds in expectation and the independent value of fractional-knapsack is lowered.

It remains to extend this result to any v . Fix v and assume without loss of generality that $v_1/\hat{c}_1 \geq \dots \geq v_n/\hat{c}_n$. Then the fractional-knapsack set function can be rewritten as

$$v_B(S) = \sum_{i \in N} (v_i/\hat{c}_i - v_{i+1}/\hat{c}_{i+1}) \min(B, \sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j)$$

and the additive set function as

$$v(S) = \sum_{i \in N} (v_i/\hat{c}_i - v_{i+1}/\hat{c}_{i+1}) (\sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j)$$

since these sums telescope.

So the ratio of independent value of $v_B(S)$ to the correlated value of $v(S)$ is minimized when the ratios of the independent value of $\min(B, \sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j)$ to the correlated value of $\sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j$ are minimized for all i . We conclude by observing that $\min(B, \sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j)$ and $\sum_{j \in S \cap \{1, \dots, i\}} \hat{c}_j$ are the fractional-knapsack set function and the additive set function when $v_i = \hat{c}_i$ over ground set $\{1, \dots, i\}$, and that their ratio is minimized when $\hat{c}_i = B/k$ for all agents i . \square

Next, we use the result from Yan [27] to bound the ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$.

LEMMA 8. *For any distribution over sets \mathcal{D} with marginal probabilities \hat{q} satisfying the ex ante budget constraint, i.e., $\sum_i \hat{c}_i \hat{q}_i \leq B$, the ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ is at least $1 - 1/\sqrt{2\pi k}$ when the market is k -large.*

PROOF. Consider the case where each agent i has cost $\hat{c}_i = B/k$ and assume that the ex ante budget constraint is satisfied, so $\sum_i \hat{q}_i \leq k$. Since any set of size at most k is feasible and since $\sum_i \hat{q}_i \leq k$, there is a distribution such that the budget constraint is always met ex post. Therefore, the correlated value of $v(\cdot)$ is equal to the correlated value of fractional-knapsack. The ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ is thus equal to the correlation gap of fractional-knapsack. Since all agents have cost B/k , the fractional-knapsack set function is equal to the k -highest-value-elements set function. By Theorem 2, the ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ is therefore $1 - 1/\sqrt{2\pi k}$.

The ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ when the ex ante budget constraint is satisfied is minimized when all agents have cost B/k by Lemma 7, so this ratio is at least $1 - 1/\sqrt{2\pi k}$. \square

We now prove the main theorem of this section which relates the approximation factor of sequential posted pricing (with ex post budget feasibility) to the optimal mechanism with ex ante budget feasibility.

THEOREM 7. *The sequential posted pricing mechanism (\hat{q}, σ) , where \hat{q} is the solution to the multilinear extension*

program (5) and where the order σ is decreasing in $\frac{v_i}{c_i}$, is a $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ approximation to the optimal mechanism in the case of regular cost distributions.

PROOF. Denote \hat{q} the optimal solution to the multilinear extension program (5). For additive value functions, linearity of expectation implies that the multilinear extension is equal to the concave closure and the optima of the multilinear extension program (5) and concave closure program (4) are the same. Their performance upper bounds that of the optimal mechanism that satisfies ex post budget feasibility by Lemma 4. The objective value of these programs with optimal solution \hat{q} is $\sum_i v_i \hat{q}_i$, which is equal to the correlated value of the additive set function $v(\cdot)$ on distributions with marginals \hat{q} . So by Lemma 8, the ratio of the independent value of fractional-knapsack to the upper bound of the optimal mechanism is at least $1 - 1/\sqrt{2\pi k}$.

The random set of agents who accept their offer in the sequential posted pricing is equal to the set of agents who are admitted by the fractional-knapsack set function on an independent random set of agents with marginals \hat{q} , without including the fractional agent. The loss from this fractional agent is at most a factor $1 - 1/k$. This posted pricing mechanism therefore has an approximation ratio of $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$. \square

As a corollary of Lemma 8, we get new correlation gap results for the fractional, and integral, knapsack set functions.

THEOREM 8. *The correlation gaps of fractional-knapsack and integral-knapsack are at least $1 - 1/\sqrt{2\pi k}$ and $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ respectively, in a k -large market.*

PROOF. We first show the correlation gap of fractional-knapsack, the correlation gap of integral-knapsack will then follow easily. We start by showing that the correlation gap is minimized when the budget constraint is satisfied. Then, we upper bound the fractional-knapsack correlated value by the correlated value of $v(\cdot)$. Finally, we apply Lemma 8.

We claim that the correlation gap of fractional-knapsack is minimized when the budget constraint is satisfied. Observe that if the budget constraint is not satisfied, then it is possible to decrease some \hat{q}_i such that the correlated value of fractional-knapsack remains the same. Since decreasing some \hat{q}_i only decreases the independent value of fractional-knapsack, the ratio of the independent value to the correlated value also decreases.

Clearly, the fractional-knapsack correlated value is upper bounded by the correlated value of $v(\cdot)$. Therefore, the correlation gap of fractional-knapsack is at least the ratio of the independent value of fractional-knapsack to the correlated value of $v(\cdot)$ when the budget constraint is satisfied, so at least $1 - 1/\sqrt{2\pi k}$ by Lemma 8.

Finally, observe that the correlated value of fractional-knapsack upper bounds the correlated value of integral-knapsack and that the independent value of integral-knapsack is a $1 - 1/k$ approximation to the independent value of fractional-knapsack. Therefore, the correlation gap of integral-knapsack is at least $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$. \square

Comparison of Sequential and Oblivious posted pricing.

We now compare the approximation ratio for additive value functions achieved using the sequential posted pricing

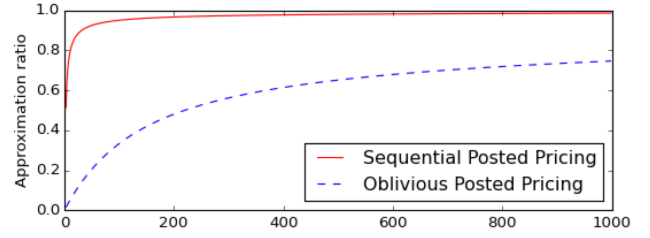


Figure 2: Comparison of the approximation ratios obtained for additive value functions by the two different approaches. On the horizontal axis is k , the size of the market.

mechanism with the bang per buck order, $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$, and using oblivious posted pricing where the budget is lowered, $(1 - \epsilon)(1 - e^{-\epsilon^2(1-\epsilon)k/12})$. Figure 2 shows that the approximation ratio with the sequential ordering approaches 1 much faster than with the oblivious ordering as the size of the market increases. To obtain these results for oblivious posted pricing, we numerically solved for the best ϵ . We emphasize that we are comparing the theoretical bounds of these approaches, and not empirical performances.

6. COMPUTING PRICES

In the two previous sections, we gave conditions under which optimal prices from the multilinear extension program (5) perform well when offered sequentially or obliviously. In this section, we consider the computational problem of finding these prices. For submodular value functions, we reduce the problem to the well-known greedy algorithm for submodular optimization. For additive value functions, we use a simple method based on the Lagrangian relaxation of the budget constraint.

6.1 The Lagrangian Relaxation for Additive Value Functions

Consider the case of additive value functions where the principal has a value v_i for each agent i and the value function is $v(S) = \sum_{i \in S} v_i$. Recall the virtual surplus program (2) from Section 3:

$$\begin{aligned} \max_{\mathbf{x}} \mathbf{E}_{\mathbf{c}}[v(\mathbf{x}(\mathbf{c}))] \\ \text{s.t. } \sum_i \mathbf{E}_{\mathbf{c}}[\phi_i(\mathbf{c}) x_i(\mathbf{c})] \leq B, \end{aligned} \quad (2)$$

which can be rewritten for additive value functions as:

$$\begin{aligned} \max_{\mathbf{q}} \sum_i \mathbf{E}_{\mathbf{c}}[v_i x_i(\mathbf{c})] \\ \text{s.t. } \sum_i \mathbf{E}_{\mathbf{c}}[\phi_i(\mathbf{c}) x_i(\mathbf{c})] \leq B. \end{aligned} \quad (6)$$

We show that the ex ante optimal mechanism can be found directly by taking the Lagrangian relaxation of the budget constraint (with parameter λ) of the following Lagrangian program:

$$\max_{\mathbf{x}} \lambda B + \sum_i \mathbf{E}_{\mathbf{c}}[(v_i - \lambda \phi_i(c_i)) x_i(\mathbf{c})]. \quad (7)$$

For any Lagrangian parameter λ , this objective can be optimized by pointwise optimizing $\sum_i (v_i - \lambda \phi_i(c_i)) x_i(\mathbf{c})$,

a.k.a., the *Lagrangian virtual surplus*. This pointwise optimization picks all the agents such that $v_i \geq \lambda \phi_i(c_i)$. If the virtual cost functions are monotone, i.e., in the so-called *regular* case, then this optimization gives a monotone allocation rule where an agent is picked whenever $c_i \leq \phi_i^{-1}(v_i/\lambda)$.

Notice that as the Lagrangian parameter increases, the payments of the agents, as represented by virtual costs, become more costly in the objective of the Lagrangian program (7). Thus, the expected payment of the mechanism is monotonically decreasing in the Lagrangian parameter. With $\lambda = 0$ the Lagrangian virtual surplus optimizer simply maximizes $v(\mathbf{x})$ and pays each agent selected the maximum cost in the support of her distribution. If this payment is under budget then it is optimal, otherwise, we can increase λ until the budget constraint is satisfied. For example, with $\lambda = \infty$ the empty set of agents is selected and no payments are made. The optimal mechanism is the one that meets the budget constraint with equality. In the case that the expected payment is discontinuous then mixing between the least over-budget and least under-budget mechanism is optimal. For further discussion of Lagrangian virtual surplus optimizers, see Devanur et al. [13].

PROPOSITION 1. *The Lagrangian virtual surplus optimizer (or appropriate mixture thereof) that meets the budget constraint with equality is the Bayesian optimal ex ante budget feasible mechanism.*

Lagrangian virtual surplus optimization suggests selecting an agent i when her private cost c_i is below $\phi_i^{-1}(v_i/\lambda)$. The mechanism that achieves this outcome posts the price of $\hat{c}_i = \phi_i^{-1}(v_i/\lambda)$ to agent i . Denote by $\hat{q}_i = F_i(\hat{c}_i)$ the probability that i accepts the price \hat{c}_i . For the prices \hat{c}_i , the total expected payments are $\sum_i \hat{c}_i \hat{q}_i$. When the virtual cost functions are monotone and strictly increasing, there is a Lagrangian parameter for which the budget constraint is met with equality, i.e., with $\sum_i \hat{c}_i \hat{q}_i = B$. The optimal ex ante mechanism is therefore the posted price mechanism that posts \hat{c}_i to each agent i for the Lagrangian parameter λ that satisfies $\sum_i \hat{c}_i \hat{q}_i = B$. Note that such a Lagrangian parameter λ can be arbitrarily well approximated since $\sum_i \hat{c}_i \hat{q}_i$ is decreasing as a function of λ .

Example 1. Consider n agents with costs drawn uniformly and i.i.d. from $[0, 1]$ and uniform additive value function $v_i = 1$ for all i , i.e., the cardinality function. The virtual cost function is $\phi(c) = c + \frac{F(c)}{f(c)} = 2c$. The Lagrangian parameter $\lambda = \frac{1}{2}\sqrt{n/B}$ induces a uniform posted price of $\hat{c} = \sqrt{B/n}$ which is accepted with probability $\hat{q} = \sqrt{B/n}$ for an expected payment of B/n . Summing over all n agents, the budget is balanced ex ante.

6.2 A Reduction to the Greedy Algorithm for Submodular Optimization

For general submodular value functions we reduce the optimization of the multilinear extension program (5), restated below, to the problem of optimizing a submodular function subject to a cardinality constraint. This problem of optimizing a submodular function under cardinality, knapsack, or matroid constraints is well studied and the *greedy algorithm* gives a $1 - 1/e$ approximation for knapsack and cardinality constraints; see Nemhauser et al. [21], Khuller et al. [18],

and Sviridenko [25].

$$\begin{aligned} \max_{\mathbf{q}} V(\mathbf{q}) \\ \text{s.t. } \sum_i q_i F_i^{-1}(q_i) \leq B. \end{aligned} \tag{5}$$

Define the *cost curve* of agent i to be the expected payment to agent i , i.e., $q_i F_i^{-1}(q_i)$ in our case. The main difference between the multilinear extension program (5) and the knapsack setting considered in the literature is that the cost curves in the knapsack setting are linear in q_i . Our reduction to the greedy algorithm is the following. We divide each agent i , called a *big agent*, in cost space into m discrete agents i_j of equal cost, called the *small agents*. An agent i_j corresponds to the j th increase of q_i , starting from $q_i = 0$, that has cost B/m . We set $1/m$ as a fraction of the total budget B which fixes the number of steps in the algorithm to be m . With large m , the reduction becomes a finer discretization.

Before formally describing the reduction, we introduce some notation. For each i and j , let δ_{ij} be the j th increase in q_i , starting from $q_i = 0$, that has cost B/m , i.e., δ_{ij} satisfying $B/m = F_i^{-1}(\sum_{k \leq j} \delta_{ik}) \cdot (\sum_{k \leq j} \delta_{ik}) - F_i^{-1}(\sum_{k < j} \delta_{ik}) \cdot (\sum_{k < j} \delta_{ik})$. Given a set S of small agents, the continuous solution corresponding to S is $\mathbf{q}(S)$ with $q_i(S) = \sum_{j: i_j \in S} \delta_{ij}$.

The reduction.

1. For each agent i , create m small agents i_j where $1 \leq j \leq m$ so that the reduced instance has mn agents.
2. For each small agent i_j , its cost is B/m .
3. For each small agent i_j , its marginal contribution $V_S(i_j)$ in value to a set S is the marginal contribution of increasing the fraction of agent i corresponding to S by δ_{ij} , i.e., $V(\mathbf{q}') - V(\mathbf{q}(S))$ where $q'_i = q_i(S) + \delta_{ij}$ and $q'_j = q_j(S)$ for $j \neq i$.

We show that the solution to the reduced problem that we obtained with the greedy algorithm for cardinality constraint corresponds to a solution for the multilinear extension program (5) that is a $1 - 1/e - o(1)$ approximation, almost matching the performance of the greedy algorithm for knapsack constraint with integral agents and linear cost curves. We start by showing that if a solution is feasible in the reduced problem, then the continuous solution corresponding to it is a feasible solution to the multilinear extension program (5). Then, with access to exact values of the increases δ_{ij} and of the marginal contributions $V_S(i_j)$, the approximation ratio is $1 - 1/e - o(1)$. Finally, in the appendix, we show that it is possible to approximate δ_{ij} and $V_S(i_j)$ with estimates that cause an additional loss of $o(1)$ to the approximation ratio.

From a set of small agents to a continuous solution for the big agents.

Previously, we defined a distribution to be regular if the virtual cost function is monotonically increasing. An alternate definition is that a distribution F is regular if the cost curve $q \cdot F^{-1}(q)$ is convex. This definition is the analogue to the revenue curve being concave for regular distributions when the agents are buyers, and not sellers, from Bulow and Roberts [7].

Recall that given a set S of small agents, the continuous solution corresponding to S is $\mathbf{q}(S)$ with $q_i(S) = \sum_{j:i_j \in S} \delta_{ij}$ and that δ_{ij} is the j th increase in q_i that has cost B/m . Therefore, given a set S of small agents of size at most m such that for any $\delta_{ij} \in S$, $\delta_{ik} \in S$ for all $k < j$, then $\mathbf{q}(S)$ has cost at most B . The condition that if $\delta_{ij} \in S$, then $\delta_{ik} \in S$ for all $k < j$, is equivalent to the condition that greedy always picks small agents corresponding to lower quantiles before small agents corresponding to higher quantiles, which we show formally.

LEMMA 9. *Given two small agents i_k and i_j such that $k < j$, the greedy algorithm with a cardinality constraint picks i_k before i_j for regular distributions F_i .*

PROOF. Since all small agents have equal cost, we need to show that i_k has a larger marginal contribution than i_j to any set S of small agents such that $i_k, i_j \notin S$. Since $V(\cdot)$ is monotone, it suffices to show that $\delta_{ik} > \delta_{ij}$. In quantile space, the cost of increasing some quantile q_i by a fix amount is increasing in q_i since $q_i \cdot F^{-1}(q_i)$ is convex by definition of regular distributions. Therefore, in cost space, the increase in quantile δ_i that is obtained by increasing the cost curve by a fix amount is decreasing, so $\delta_{ik} > \delta_{ij}$. \square

The case of irregular distributions is considered in Appendix B.

With exact values of δ_{ij} and $V_S(i_j)$.

We consider the case where the exact values of the increases in \mathbf{q} and marginal contributions are given by an oracle. We show that finding a good solution to this reduced problem with small agents gives us a good solution to the problem with big agents.

LEMMA 10. *The optimal solution S^* to the reduced problem satisfies $V(\mathbf{q}(S^*)) \geq (1 - o(1))V(\hat{\mathbf{q}})$ where $\hat{\mathbf{q}}$ is the optimal solution to the multilinear extension program (5).*

PROOF. We pick the step size to be $m = n^2$. The proof shows that there exists a set S that is close to a feasible solution in the reduced problem and such that $\mathbf{q}(S)$ is a better solution than $\hat{\mathbf{q}}$. Let S be the set of small agents such that $\mathbf{q}(S)$ is maximized subject to $\mathbf{q}(S) \leq \hat{\mathbf{q}}$. Define S^{+1} to be the set containing all small agents in S and one additional small agent for each big agent i . Observe that $V(\mathbf{q}(S^{+1})) \geq V(\hat{\mathbf{q}})$ since $V(\cdot)$ is non-decreasing. So there is a feasible solution to the discretized problem such that if we add one small agent for each big agent i , then we obtain a better solution than the optimal solution to the original problem.

Greeditly remove agents by minimal marginal contribution from S^{+1} until we get a feasible solution S . The number of small agents who need to be removed is n since S is feasible. Since S contains n^2 small agents, by the greediness and the fact $V(\cdot)$ is concave along any line of positive direction, $(1 + 1/n)V(\mathbf{q}(S)) \geq V(\mathbf{q}(S^{+1}))$.

Therefore,

$$\begin{aligned} (1 + o(1))V(\mathbf{q}(S^*)) &\geq (1 + o(1))V(\mathbf{q}(S)) \\ &\geq V(\mathbf{q}(S^{+1})) \\ &\geq V(\hat{\mathbf{q}}). \end{aligned}$$

\square

Next, we show that the reduced problem can be optimized.

LEMMA 11. *Let S be the set returned by the greedy algorithm for submodular functions under a cardinality constraint on the reduced problem, then*

$$V(\mathbf{q}(S)) \geq (1 - 1/e)V(\mathbf{q}(S^*))$$

where S^* is the optimal solution to the reduced problem.

PROOF. Observe that the objective function in the reduced problem is a submodular function. This follows directly from the concavity of $V(\cdot)$ along any positive line of direction. In addition, since all small agents have cost B/m , the constraint is a cardinality constraint. Since the greedy algorithm for submodular functions under a cardinality constraint is a $1 - 1/e$ approximation for submodular functions, we get the desired result. \square

We now have the tools to show that if we had an oracle for the increases and marginal contributions, the greedy algorithm on the reduced instance would give us a $1 - 1/e - o(1)$ approximation.

LEMMA 12. *Let S be the output of the greedy algorithm on the reduced instance, where exact values of δ_{ij} and $V_S(i_j)$ are given by an oracle at each iteration, then $V(\mathbf{q}(S)) \geq (1 - 1/e - o(1))V(\hat{\mathbf{q}})$, where $\hat{\mathbf{q}}$ is the optimal solution to the multilinear extension program (5).*

PROOF. We combine the results from the discretization that causes a $o(1)$ loss with the greediness of the algorithm that is a $1 - 1/e$ approximation to obtain the desired result.

By Lemma 11 and Lemma 10,

$$V(\mathbf{q}(S)) \geq (1 - 1/e)V(\mathbf{q}(S^*)) \geq (1 - 1/e - o(1))V(\hat{\mathbf{q}})$$

where S^* is the optimal solution to the reduced problem. \square

The analysis which shows that it is possible to approximate δ_{ij} and $V_S(i_j)$ with estimates that cause an additional loss of $o(1)$ to the approximation ratio is deferred to the appendix.

7. CONCLUSION

We consider questions of budget feasibility in a Bayesian setting. We show that simple posted pricing mechanisms are ex post budget feasible and approximate the Bayesian optimal mechanism. Our analysis first considers the ex ante relaxation where the budget constraint is allowed to hold in expectation. Good approximations are obtained when this ex ante relaxation is optimized for a slightly reduced budget or when the agents are ordered by bang-per-buck (value divided by offered price). The latter approach, in the case of additive value functions when it applies, gives better bounds.

Another method for designing posted pricing mechanisms from the literature comes from the generalized magician's problem from Alaei [2]. Unfortunately, this approach does not satisfy the monotonicity property of Theorem 4 needed to apply known results that give a good approximation in the case of submodular functions. Thus, it is unclear whether this approach can be adapted to budget feasibility questions.

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APPENDIX

A Symmetric Costs and Values

In this section we study symmetric environments where both the distribution of costs and the value function are symmetric. A submodular value function is symmetric if the value of a set only depends on the cardinality of that set, i.e., $v(S) = g(|S|)$ for some function $g(\cdot)$. In this setting, we obtain an oblivious posted pricing that achieves an approximation ratio of $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ where k is the size of the market, which is identical to the approximation obtained in the additive case with sequential posted pricing. We assume that the distribution of costs is regular.

The following technicalities are used for this section only. We overload the notation and denote by $v(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the concave hull of the points $\{(i, v(S_i))\}_{i=0}^n$ where S_i is any set of size i . The posted prices in this section are symmetric and are defined by a single price \hat{c} , i.e., $\hat{c} = (\hat{c}, \dots, \hat{c})$ and $\hat{q} = (\hat{q}, \dots, \hat{q})$. Note that the market size k in such a symmetric setting is $k = B/\hat{c}$.

We start with two lemmas that highlight symmetric properties of the optimal solution to the concave closure program in this symmetric setting.

LEMMA 13. *For symmetric submodular value function $v(\cdot)$ and symmetric distributions of costs, the optimal solution \hat{q} to the concave closure program (4) is symmetric, i.e., $\hat{q}_i^+ = \hat{q}_j^+$ for all i, j .*

PROOF. By the concavity of the concave closure and the convexity of cost curves (since the distribution of costs is regular), the program we wish to optimize is symmetric and convex, so the optimal solution is symmetric. \square

LEMMA 14. *For symmetric monotone submodular value function $v(\cdot)$ and symmetric distributions of costs, there exists a distribution \mathcal{D} over sets of agents with marginals $\hat{q}^+ = (\hat{q}^+, \dots, \hat{q}^+)$ such that $\mathbf{E}_{S \sim \mathcal{D}}[v(S)] = V^+(\hat{q}^+)$ and such that all sets S and T that can be drawn from \mathcal{D} have size either $\lfloor k \rfloor$ or $\lceil k \rceil$.*

PROOF. First, note that $B = \hat{c} \cdot n \cdot \hat{q}^+$ since \hat{q}^+ is the optimal solution to the concave closure program and since $v(\cdot)$ is monotone, which implies that $k = n \cdot \hat{q}^+$ since $k = B/\hat{c}$.

The expected value of a set of expected size $n \cdot \hat{q}^+$ drawn from a distribution is at most $v(n \cdot \hat{q}^+)$ by the definition of concave hull. By taking a distribution \mathcal{D} that is a mixture of sets of size $\lfloor n \cdot \hat{q}^+ \rfloor = \lfloor k \rfloor$ and $\lceil n \cdot \hat{q}^+ \rceil = \lceil k \rceil$ such that the marginals are \hat{q}^+ , the expected value of a set drawn from \mathcal{D} is $v(n \cdot \hat{q}^+)$ since $v(S)$ is submodular. Combining the two previous observations, $\mathbf{E}_{S \sim \mathcal{D}}[v(S)] = V^+(\hat{q}^+)$ since the concave closure is the maximum expected value over distributions with some marginals \hat{q} . \square

Given quantiles $\hat{q} = (\hat{q}, \dots, \hat{q})$, the value of the concave closure $V^+(\hat{q})$ can be computed easily by Lemma 14 and symmetry. The concave closure program can therefore be approximated arbitrarily well and efficiently when there is symmetry, by using binary search to get arbitrarily close to the optimal quantile \hat{q} . Our approach for obtaining the desired approximation is to construct an additive function that lower bounds the symmetric submodular function on feasible sets and that upper bounds it otherwise.

THEOREM 9. *In the case of symmetric monotone submodular value functions and symmetric regular cost distributions, the oblivious posted pricing $\hat{c} = (\hat{c}, \dots, \hat{c})$ with $\hat{c} =$*

$F^{-1}(\hat{q}^+)$ is an $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ approximation to the optimal ex ante mechanism, where $\hat{q}^+ = (\hat{q}^+, \dots, \hat{q}^+)$ is the optimal solution to the concave closure program (4) and k is the size of the market.

PROOF. By Lemma 14, there exists a distribution \mathcal{D} over sets of agents with marginals \hat{q}^+ such that $\mathbf{E}_{S \sim \mathcal{D}}[v(S)] = V^+(\hat{q}^+)$ and such that sets drawn from \mathcal{D} have size $\lfloor k \rfloor$ or $\lceil k \rceil$. We consider the additive value function $v^{add}(\cdot)$ defined as follow:

$$v^{add}(S) = |S| \frac{v(\lfloor k \rfloor)}{\lfloor k \rfloor}$$

and overload the notation for $v^{add}(\cdot)$ similarly as for $v(\cdot)$. We make the following observations about $v^{add}(\cdot)$:

- $v^{add}(i) \leq v(i)$ for $i \leq \lfloor k \rfloor$ and $v^{add}(i) \geq v(i)$ otherwise, by submodularity.
- $\mathbf{E}_{S \sim \mathcal{D}}[v^{add}(S)] \geq \mathbf{E}_{S \sim \mathcal{D}}[v(S)]$, since $v^{add}(\lceil k \rceil) \geq v(\lceil k \rceil)$ and $v^{add}(\lfloor k \rfloor) = v(\lfloor k \rfloor)$.
- $v(\cdot)$ is an additive set function with values $v_i = \frac{1}{\lfloor k \rfloor} v(\lfloor k \rfloor)$ for each element.

Since the feasible sets are sets of size at most $\lfloor k \rfloor$ and by the first observation on $v^{add}(\cdot)$, the performance of the posted pricing mechanism is at least the independent integral knapsack value of $v^{add}(\cdot)$. The independent integral knapsack value of $v^{add}(\cdot)$ is at most a factor $(1 - 1/k)$ away from $\mathbf{E}_{S \sim \hat{q}^+}[v^{add}(S)]$, its independent fractional knapsack value. By Lemma 8 and the third observation on $v^{add}(\cdot)$, $\mathbf{E}_{S \sim \hat{q}^+}[v^{add}(S)] \geq (1 - \frac{1}{\sqrt{2\pi k}}) \mathbf{E}_{S \sim \mathcal{D}}[v^{add}(S)]$. By the second observation on $v^{add}(\cdot)$, $\mathbf{E}_{S \sim \mathcal{D}}[v^{add}(S)] \geq V^+(\hat{q}^+)$. Since $V^+(\hat{q}^+)$ is an upper bound on the optimal mechanism by Lemma 4, we get the desired result. \square

Note that in previous settings, we used the solution to the multilinear extension program to define the posted pricing mechanisms. In this setting, we used the solution to the concave closure program in order to take advantage of the concavity of the objective function for computational purposes. Finally, note that in the symmetric case, sequential posted pricing offers no advantage compared to oblivious posted pricing.

B Irregular Distributions

In this section, we consider irregular distributions. Recall that a distribution F is regular if the virtual cost function is increasing, or equivalently, if the cost curve $q \cdot F^{-1}(q)$ is convex. The ironing method introduced by Myerson [19] gives monotone ironed virtual costs and convex cost curves. With these convex cost curves, we construct *randomized* posted pricing mechanisms that enjoy the same approximation ratios as the deterministic mechanisms, albeit with a generalized definition of the market size k for randomized posted pricings. Additionally, in the case of additive objective functions, the sequential posted pricing is derandomized.

Denote the cost curve of agent i by $C_i(q_i) = q_i F_i^{-1}(q_i)$. Bulow and Roberts [7] observed that the derivative of the cost curve with respect to quantile is equal to the virtual cost function, $C'_i(q_i) = \phi_i(c_i)$. The ironing method constructs the convex hull $\bar{C}_i(q_i)$ of the cost curve $C_i(\cdot)$. For $q_i = F_i(c_i)$, the ironed virtual costs are $\bar{\phi}_i(c_i) = \bar{C}'_i(q_i)$. By taking the convex hull of the cost curves, we have convex

cost curves and monotone ironed virtual costs as desired. The next two lemmas show that expected payments $\bar{C}_i(\hat{q}_i)$ are feasible while serving each agent with probability \hat{q}_i , and that no incentive compatible mechanism can do better.

LEMMA 15. [Myerson [19], Bulow and Roberts [7]] For any agent with cost drawn from distribution F_i and any incentive compatible mechanism that selects agent i with probability \hat{q}_i , the expected payment to agent i is at least $\bar{C}_i(\hat{q}_i)$.

We give the proof of the following known lemma since it exhibits how to pick the prices and the probabilities of the randomized mechanisms.

LEMMA 16. [Myerson [19]] Expected payment $\bar{C}_i(\hat{q}_i)$ while serving agent i with probability \hat{q}_i is achievable using a randomized posted pricing with at most two prices.

PROOF. Fix a seller i and an ex ante sale probability \hat{q}_i . If $\hat{q}_i = C_i(\hat{q}_i)$, then it suffices to post price $F_i^{-1}(\hat{q}_i)$. Otherwise, let a be the largest quantile smaller than \hat{q}_i such that $\bar{C}_i(a) = C_i(a)$. Similarly, let b be the smallest quantile larger than \hat{q}_i such that $\bar{C}_i(b) = C_i(b)$. The interval $[a, b]$ corresponds to the ironed interval in which \hat{q}_i falls in. By the definition of convex hull, we get

$$\begin{aligned}\bar{C}_i(\hat{q}_i) &= \left(1 - \frac{\hat{q}_i - a}{b - a}\right)\bar{C}_i(a) + \left(1 - \frac{b - \hat{q}_i}{b - a}\right)\bar{C}_i(b) \\ &= \left(1 - \frac{\hat{q}_i - a}{b - a}\right)C_i(a) + \left(1 - \frac{b - \hat{q}_i}{b - a}\right)C_i(b).\end{aligned}$$

Therefore, posting price $F_i^{-1}(a)$ with probability $1 - \frac{\hat{q}_i - a}{b - a}$ and $F_i^{-1}(b)$ with probability $1 - \frac{b - \hat{q}_i}{b - a}$ has expected payment $\bar{C}_i(\hat{q}_i)$ and the ex ante probability that seller i accepts the price is \hat{q}_i . \square

By Lemma 15 and Lemma 16, the ex ante results also hold for the irregular case using randomized posted pricing. The following definition generalizes the notion of posted prices to allow for randomization.

Definition 6. For a randomized posted pricing \hat{q} ,

- Prices \hat{c}_{i1} and \hat{c}_{i2} with probabilities of picking each price are induced by \hat{q}_i .
- Randomly pick \hat{c}_{i1} or \hat{c}_{i2} .
- In the case of sequential posted pricing, set the ordering to be in decreasing order of bang-per-buck.

Definition 7. With randomized posted pricing, a market is k -large if $B/\hat{c}_{ij} \geq k$ for all agents i and $j \in \{1, 2\}$.

B.1 From Ex Ante to Ex Post with Additive Value Functions

For the additive case, we first show that the ex post randomized posted pricing performs well and then derandomize the mechanism.

THEOREM 10. The randomized sequential posted pricing mechanism $(\hat{q}, \sigma(\cdot))$ that serve agents with probability \hat{q} , where \hat{q} is the solution to the multilinear extension program (5) and where the order $\sigma(\cdot)$ is decreasing in $\frac{v_i}{\hat{c}_i}$, is a $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ approximation to the optimal mechanism in a k -large market.

PROOF. We start by showing that the randomized sequential posted pricing performs better than a deterministic sequential posted pricing with the same ex ante performance and a market that is k -large. Consider a randomized agent i who is offered $\hat{c}_i = \hat{c}_{i1}$ with probability ρ and $\hat{c}_i = \hat{c}_{i2}$ otherwise. Remove agent i and replace it with two deterministic agents $i1$ and $i2$ with value v_i , who are offered \hat{c}_{i1} and \hat{c}_{i2} and who accept their price with probability $\rho F_i(\hat{c}_{i1})$ and $(1 - \rho)F_i(\hat{c}_{i2})$ respectively. Call this new posted pricing the deterministic instance and the original posted pricing the randomized instance.

Both instances have the same ex ante performance since the expected total cost remains the same and since agent i accepts his offer with probability equal to the sum of the probabilities that agents $i1$ and $i2$ accept their offer. Fix a set S of agents who accept their offer that does not include i and fix these offers. Notice that in both the randomized and deterministic instance, there is an expected increase in the total cost of $\hat{c}_{i1}\rho F_i(\hat{c}_{i1}) + \hat{c}_{i2}(1 - \rho)F_i(\hat{c}_{i2})$ caused by agent i to S . However, in the randomized instance, this increase in cost is either \hat{c}_{i1} or \hat{c}_{i2} and in the deterministic instance, this increase in cost can also be $\hat{c}_{i1} + \hat{c}_{i2}$. Since agents are ordered by decreasing bang-per-buck, the loss from agents that do not fit in the ex post budget constraint is greater in the deterministic case. Therefore, the loss of the fractional knapsack value with respect to the ex ante performance of the mechanism is greater in the deterministic instance.

Now note that this argument can be repeated inductively until all the agents left are deterministic. So the approximation ratio obtained by the randomized mechanism is $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$, by combining Lemma 8 and the $1 - 1/k$ loss from dropping the fractional agent. \square

We now show that the mechanism can be derandomized.

THEOREM 11. Any sequential randomized posted pricing $(\hat{q}, \sigma(\cdot))$ can be modified into a sequential deterministic posted pricing in the case of additive value functions.

PROOF. The proof proceeds in two steps. The first reduces the number of randomized agents until there is one left by using properties of ironed intervals. The second step is to simply pick the best of the two prices that are offered to the last randomized agent.

Consider a randomized posted pricing $(\hat{q}, \sigma(\cdot))$ with at least two agents i and j that are randomized. The marginal cost per unit value of these two agents are $\bar{C}'_i(\hat{q}_i)/v_i = \bar{\phi}'_i(c_i)/v_i$ and $\bar{\phi}'_j(c_j)/v_j$. Without loss of generality, assume $\bar{\phi}'_i(c_i)/v_i \leq \bar{\phi}'_j(c_j)/v_j$. Since both of these agents are randomized, \hat{q}_i and \hat{q}_j are within ironed intervals and their ironed virtual costs are constants within these intervals. With no loss in the objective function, we can therefore increase \hat{q}_i and decrease \hat{q}_j such that the budget still binds and such that either \hat{q}_i or \hat{q}_j is at the extremity of the ironed interval it is in, and therefore not randomized anymore. This construction can be repeated until one randomized agent is left.

Consider a randomized posted pricing with a unique randomized agent i who is offered $\hat{c}_i = \hat{c}_{i1}$ with probability ρ and $\hat{c}_i = \hat{c}_{i2}$ otherwise. The proof of Theorem 10 shows that the ratio between the performance of the optimal mechanism and the expected fractional knapsack value is at least $1 - 1/\sqrt{2\pi k}$. Agent i is either offered \hat{c}_{i1} or \hat{c}_{i2} , so by expectations, with at least one of these two offers, the previous

ratio is at least $1 - 1/\sqrt{2\pi k}$. Dropping the fractional agent and keeping the best price to offer to agent i , we therefore get a $(1 - 1/\sqrt{2\pi k})(1 - 1/k)$ approximation for a deterministic mechanism. \square

COROLLARY 2. *Any sequential randomized posted pricing $(\hat{q}, \sigma(\cdot))$ can be modified with high probability into a sequential deterministic posted pricing in the case of additive value functions with an additional $o(1)$ loss in polynomial time.*

PROOF. We need to compute which offered price between \hat{c}_{i1} and \hat{c}_{i2} performs better in terms of fractional knapsack value. Fractional-knapsack is a submodular function and the multilinear extension of submodular functions can be approximated arbitrarily well by sampling using Chernoff bounds. Therefore, with high probability, it is possible to compare arbitrarily well the fractional knapsack value obtained with the two offered prices to agent i . \square

B.2 From Ex Ante to Ex Post with Submodular Value Functions

With submodular value functions, the analysis for the oblivious randomized posted pricing is identical as the analysis for the oblivious deterministic posted pricing. In Section 4, Theorem 5 shows that by lowering the budget by some small amount, we get that the sum of the costs does not exceed the budget with high probability. Note that this results does not only hold for deterministic agents but also for randomized agents since the payment p_i to an agent i only need to be bounded by B/k and is not restricted to be either 0 or \hat{c}_i . Therefore, the sum of the costs does not exceed the budget with high probability in the randomized case as well and the remaining of the analysis of section 4 also holds.

THEOREM 12. *For $\epsilon \in (0, 1/2)$, if the randomized oblivious posted pricing \hat{q} , where \hat{q} is the optimal solution to the multilinear extension program (5) with budget $(1 - \epsilon)B$, satisfies $2/\epsilon \leq k \leq B/\max_i \hat{c}_i$, then this posted pricing mechanism is a $(1 - 1/e)(1 - \epsilon)(1 - e^{-\epsilon^2(1 - \epsilon)^{k/12}})$ approximation to the optimal mechanism for submodular value functions and $(1 - \epsilon)(1 - e^{-\epsilon^2(1 - \epsilon)^{k/12}})$ for additive value functions.*

C Computing prices with estimates of δ_{ij} and $V_S(i_j)$

We show that we can use the greedy algorithm with estimates of the increases and the marginal contributions, that we can compute. Let $\tilde{q}(S)$ be defined similarly to $q(S)$ but with estimates $\tilde{\delta}_{ij}$. The first lemma shows that the value of the optimal solution to the reduced problem has almost the same value as when the increases δ_{ij} are estimated. The second lemma extends Lemma 11 to the case where greedy is run with estimated marginal contributions $\tilde{V}_S(i_j)$ and any $\tilde{\delta}_{ij}$.

LEMMA 17. *Let S^* be the optimal solution to the reduced problem with exact value of δ_{ij} and $V_S(i_j)$, then $V(\tilde{q}(S^*)) \geq (1 - o(1))V(q(S^*))$.*

PROOF. We need to find the increase satisfies $B/m = F_i^{-1}(\sum_{k \leq j} \delta_{ik}) \cdot (\sum_{k \leq j} \delta_{ik}) - F_i^{-1}(\sum_{k < j} \delta_{ik}) \cdot (\sum_{k < j} \delta_{ik})$. To approximate it, we find $\tilde{\delta}_{ij}$ such that $(1 - 1/n^3)\delta_{ij} \leq \tilde{\delta}_{ij} \leq \delta_i$, which can be done easily since the weight functions are increasing.

Recall that $\tilde{q}(S)$ is defined similarly to $q(S)$ but with estimates $\tilde{\delta}_{ij}$. Let S^* be the optimal solution of the problem with small agents without noise. Since $(1 - 1/n^3)\delta_{ij} \leq \tilde{\delta}_{ij} \leq \delta_{ij}$ for all i, j , we get that $\tilde{q}(S^*) \geq (1 - 1/n)q(S^*)$. By the concavity of $V(\cdot)$ along positive lines of direction, we get that $V(\tilde{q}(S^*)) \geq (1 - 1/n)V(q(S^*))$. \square

LEMMA 18. *Let \tilde{S} be the set returned by the greedy algorithm on the reduced problem with estimates $\tilde{\delta}_{ij}$ and $\tilde{V}_S(i_j)$, then $V(\tilde{q}(\tilde{S})) \geq (1 - 1/e - o(1))V(\tilde{q}(S^*))$ w.h.p., where S^* is the optimal solution to the reduced problem with estimates $\tilde{\delta}_{ij}$ and exact values $V_S(i_j)$.*

PROOF. First note that the objective function for the reduced instance is a submodular function regardless of the values of $\tilde{\delta}_{ij}$. So since we are comparing ourselves with $\tilde{q}(S^*)$, it remains to show that the greedy algorithm with a noisy oracle on marginal contribution of agents performs well.

Let $g(\cdot)$ be the objective function of the reduced instance. The marginal contributions are estimated by taking $\frac{10}{\delta^4}(1 + \ln n)$ samples of the random set with independent marginal probabilities q . By using basic Chernoff bounds as in Calinescu et al. [9], we get that with high probability, all the estimates that are computed during the algorithm have an additive error of at most $\delta^2 g(S^*)$.

Let S be the set of small agents returned by the algorithm. Let $S_i = \{e_1, \dots, e_i\}$ be the value of S after i iterations. Now since $g(\cdot)$ is submodular,

$$g(S^*) \leq g(S_{i-1}) + \sum_{e \in S^* \setminus S_{i-1}} g_{S_{i-1}}(e)$$

By the greediness of the algorithm, $\tilde{g}_{S_{i-1}}(e_i) \geq \tilde{g}_{S_{i-1}}(e)$ for all $e \in S^* \setminus S_{i-1}$. So, $g_{S_{i-1}}(e_i) + 2\delta^2 g(S^*) \geq g_{S_{i-1}}(e)$, and

$$\begin{aligned} g(S^*) &\leq g(S_{i-1}) + \frac{1}{\delta}(g_{S_{i-1}}(e_i) + 2\delta^2 g(S^*)) \\ (1 - 2\delta)g(S^*) &\leq g(S_{i-1}) + \frac{1}{\delta}g_{S_{i-1}}(e_i) \end{aligned}$$

Then, by following identically the remaining of the proof for the $\epsilon/(e - 1)$ approximation for greedy subject to a cardinality constraint, but by replacing $g(S^*)$ by $(1 - 2\delta)g(S^*)$, we get that $(1 - 1/e)g(S) \geq (1 - 2\delta)g(S^*)$, which concludes the proof. \square

Combining the previous results, we obtain the main result of this section.

THEOREM 13. *Let \tilde{S} be the output by the greedy algorithm on the reduced instance with estimates of δ_{ij} and $V_S(i_j)$, then $V(\tilde{q}(\tilde{S})) \geq (1 - 1/e - o(1))V(\hat{q})$ w.h.p., where \hat{q} is the optimal solution to the multilinear extension program (5).*

PROOF. This proof follows similarly to the one for Lemma 12, the difference is that this proof adds the loss from the estimates.

By Lemma 17 and Lemma 18,

$$V(\tilde{q}(\tilde{S})) \geq (1 - 1/e - o(1))V(\tilde{q}(S^*)) \geq (1 - 1/e - o(1))V(q(S^*))$$

where S^* is the optimal solution to the reduced problem. Using Lemma 10 that connects the discretized reduced instance to the original continuous problem, we conclude that

$$V(\tilde{q}(\tilde{S})) \geq (1 - 1/e - o(1))V(q(S^*)) \geq (1 - 1/e - o(1))V(\hat{q}).$$

\square

Note that in the case of additive value functions, the greedy algorithm is optimal when the optimization is subject to a cardinality constraint and the marginal contributions can be computed exactly. We therefore get the following result.

LEMMA 19. *Assume $v(\cdot)$ is an additive value function. Let S be the set returned by the greedy algorithm on the reduced problem with estimates $\tilde{\delta}_{i_j}$, then $V(\tilde{\mathbf{q}}(S)) \geq (1 - o(1))V(\hat{\mathbf{q}})$ w.h.p., where $\hat{\mathbf{q}}$ is the optimal solution to the multilinear extension program (5).*

Therefore, all the results in previous sections suffer an extra $1 - 1/e - o(1)$ factor in the general case of submodular value function and an extra $1 - o(1)$ factor in the case of additive value function that are due to computational constraints.